SEQUENTIAL TESTING AND CONFIDENCE INTERVALS FOR THE MTBF OF SYSTEMS HAVING EXPONENTIAL DISTRIBUTION OF THE INTERFAILURE TIMES

by

S. Zacks

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THE GEORGE WASHINGTON UNIVERSITY
School of Engineering and Applied Science
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1. Introduction

The application of the Wald SPRT to life testing of equipment, having an exponential distribution of the interfailure times, has been in practice from the fifties. ~~(see Zacks [1971; pp. 466])~~. The problem discussed in the present paper is that of the exact computation of the distributions of the stopping times, and the determination of confidence limits for the meantime between failures (MTBF), Theta, θ, after stopping. More specifically, we consider a life testing, or reliability acceptance test, in which n identical systems are put on test. Whenever a failure occurs it is instantaneously repaired, the time of failure is recorded, and the system continues to be tested. The total number of failures of the n systems, in the time interval $(0,t]$, is a Poisson process, $X_n(t)$, with mean nt/θ, where θ is the MTBF. In Section 2 we present a modified Wald SPRT, for the determination of the stopping time, and the decision of whether to accept or reject the hypotheses H_0: $\theta \geq \theta_0$ vs. H_1: $\theta \leq \theta_1$, for some $0 < \theta_1 < \theta_0 < \infty$. In Section 3

we define the distributions of the stopping times at acceptance or at rejection, and the distribution of the actual stopping time. In Section 4 we show how one can determine confidence intervals for θ after stopping. The formulae on which the algorithm for the determination of the distribution of the stopping times, the acceptance probabilities and average test length are given in Sections 5 and 6. A BASIC program for the computations is given in the Appendix.

The problem of determining confidence intervals after sequential stopping is discussed by Siegmund (1978, 1985) and Wijsman (1981). Siegmund shows how one can apply results from renewal theory to obtain approximations to the required distributions. By utilizing the specific properties of the Poisson process, as a non-decreasing process, of unit jumps, we have developed recursive equations for exact computations. The algorithm is practical with the available modern equipment. It can be easily executed with a desk top computer.

2. Stopping Boundaries for Reliability Testing

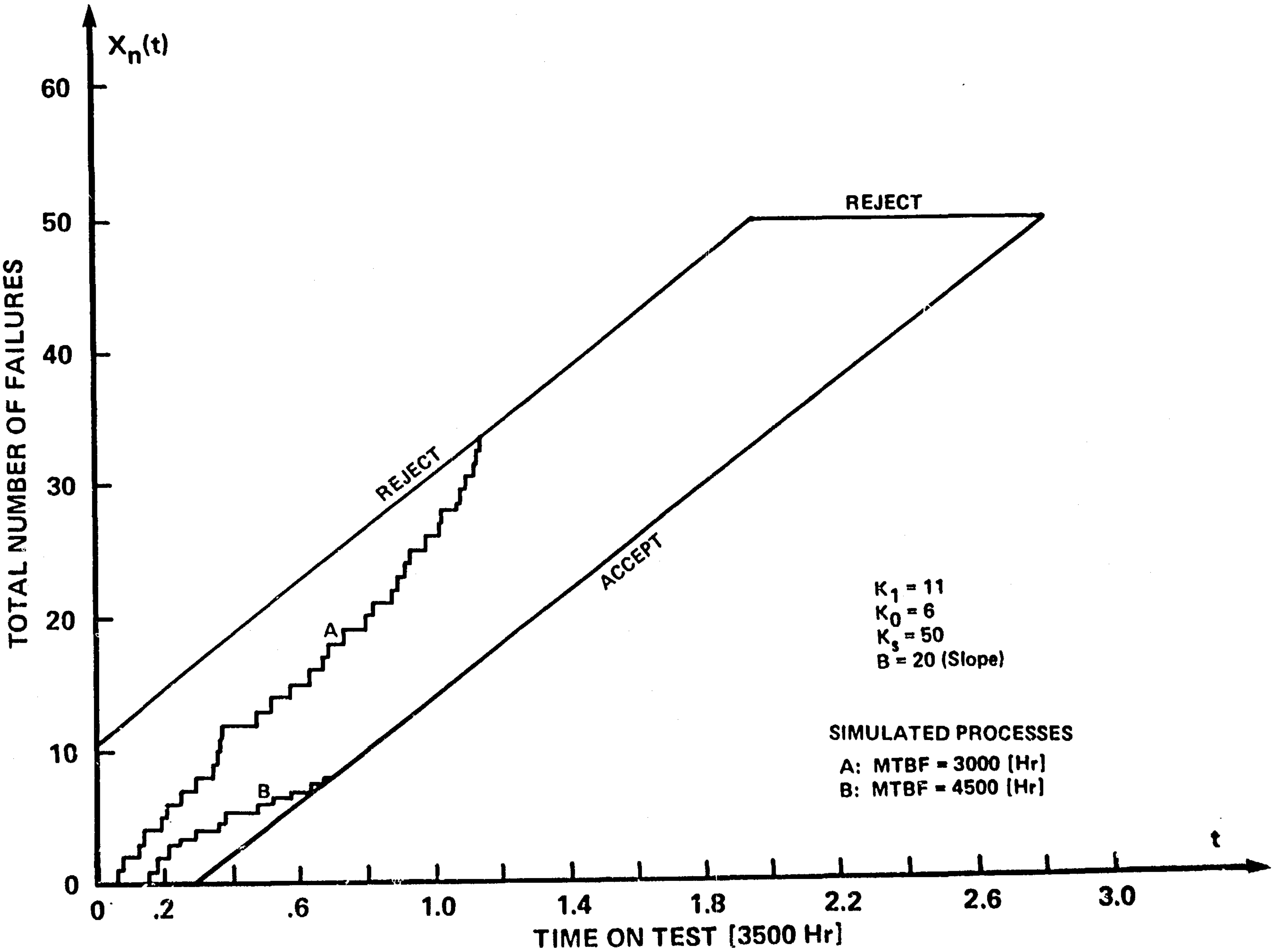
In Figure 1 we present the stopping boundaries of a modified SPRT. The modification of the SPRT is in the introduction of an upper bound, k_s, for the number of allowed failures. We assume that $k_s \geq k_0 + k_1$. Let $X_n(t)$ denote the total number of observed failures of n units on test, in the time interval $(0,t]$. The rule is that the experiment is terminated when the process, $X_n(t)$, either crosses the upper (rejection) boundary, or intersects the lower (acceptance) boundary. The upper (rejection) boundary is given by

$$b_R(t) = \begin{cases} k_1 + bt, & \text{if } 0 < t < t^* \\ k_s, & \text{if } t^* \leq t \leq t^{**} \end{cases} \tag{2.1}$$

where k_1 is a positive (integer) intercept, and $t^* = (k_s - k_1)/b$. The slope b is determined to achieve certain characteristics, which will be discussed in the next section. The lower boundary for acceptance is

Figure 1

STOPPING BOUNDARIES FOR SEQUENTIAL TESTING



given by the line

$$b_A(t) = -k_0 + bt, \; 0 < t \le t^{**}; \tag{2.2}$$

where $t^{**} = (k_s + k_0)/b$, and k_0 is an integer. The value of $t_{k_0} = k_0/b$ is the minimal time required for acceptance. By increasing the value of k_0 (with a fixed slope b) we defer the minimal acceptance time and thus decrease the probability of accepting the product. By increasing the value of k_1 we decrease the probability of rejecting the product, and by increasing the value of the slope b we increase the probability of accepting the product.

3. Distribution of Stopping Times and the Characteristics of the Sequential Procedure

We distinguish between two stopping variables, stopping while crossing the acceptance boundary, and stopping while crossing the rejection boundary. We denote the stopping time at acceptance by τ_A, and the other one by τ_R. The life testing terminates at time epoch τ, which is the smaller of τ_A and τ_R, i.e., $\tau = \min\{\tau_A, \tau_R\}$. Obviously, either $\tau = \tau_A$ or $\tau = \tau_R$. Let $F_\theta^A(t) = P_\theta\{\tau_A \le t\}$ be the cumulative distribution (CDF) of τ_A, when the MTBF is θ. Let $F_\theta^R(t)$ be the corresponding CDF of τ_R, and $F_\theta(t)$ that of τ. Since one terminates either with τ_A or with τ_R,

$$F_\theta(t) = F_\theta^A(t) + F_\theta^R(t), \tag{3.1}$$

and since $\tau \le t_s$, where $t_s = t^{**} - 1/b$, $F_\theta(t_s) = 1$. On the other hand $F_\theta^A(t_s) < 1$ and $F_\theta^R(t_s) < 1$. Notice also that $t_{k_0} \le \tau_A$, and τ_A can accept only the values $t_{k_0+i} = t_{k_0} + i/b$, $i=0,1,2,\ldots,k_s - 1$. Accordingly, $F_\theta^A(t) = 0$ for all $t < t_{k_0}$. The distribution of τ_A is discrete, with jumps at t_{k_0+i}, of size

$$P_\theta^A(i) = P_\theta\{\tau_A = t_{k_0+i}\}, \quad i = 0,\ldots,k_s - 1. \tag{3.2}$$

The value of $F_\theta^A(t)$ at $t_{k_0+k_s-1} = t_{k_0} + (k_s - 1)/b$ is the probability of acceptance. We denote this probability by $\pi(\theta)$. The distribution of τ_R, $F_\theta^R(t)$, on the other hand is continuous. Explicit formulae for $F_\theta^R(t)$ and $F_\theta^A(t)$ are given in Section 5. In Table 1 we present numerical values of these distributions, for the case of $N = 20$, $k_0 = 6$, $k_1 = 11$, $k_s = 50$, $b = 20$ and $\theta = 3{,}500$ [hr]. We see in this table that the acceptance probability with the specified boundaries is $\pi(3{,}500) = .4121$. The 10-th percentile of $F_\theta(t)$ is $\tau_{.10} = .75 \times 3{,}500 = 2{,}625$ [hr]. The median is $\tau_{.50} = 2.13 \times 3{,}500 = 7{,}455$ [hr] and the 90-th percentile is $\tau_{.90} = 2.55 \times 3{,}500 = 8{,}925$ [hr]. The expected stopping time, or average test length (ATL) can be easily computed by the approximation formula

$$ATL(\theta) = \frac{1}{b} \sum_{i=1}^{k_0+k_s-1} (i - \tfrac{1}{2})[F_\theta(t_i) - F_\theta(t_{i-1})], \tag{3.3}$$

where $t_i = i/b$, $i = 1,2,\ldots,k_s + k_0 - 1$, and $t_0 \equiv 0$. A formula for the exact computation of $ATL(\theta)$ is given in Section 6. In Figure 2 we present the graphs of the acceptance probability function, $\pi(\theta)$, and the functions $ATL(\theta)$ and the median stopping time $\tau_{.5}(\theta)$.

We see that the average test length attains a maximum value of 1.9 [3,500 hr] when $\theta = 3{,}500$ [hr]. On the other hand, when $\theta \le 2{,}500$ [hr] or $\theta \ge 4{,}500$ [hr] then $ATL(\theta) < 3{,}500$ [hr], which is almost 50% savings in total time on test. There is some discrepancy between $ATL(\theta)$ and $\tau_{.5}(\theta)$. When $\theta < 2{,}500$ [hr] or $\theta > 3{,}750$ [hr], $ATL(\theta) > \tau_{.5}(\theta)$.

Table 1

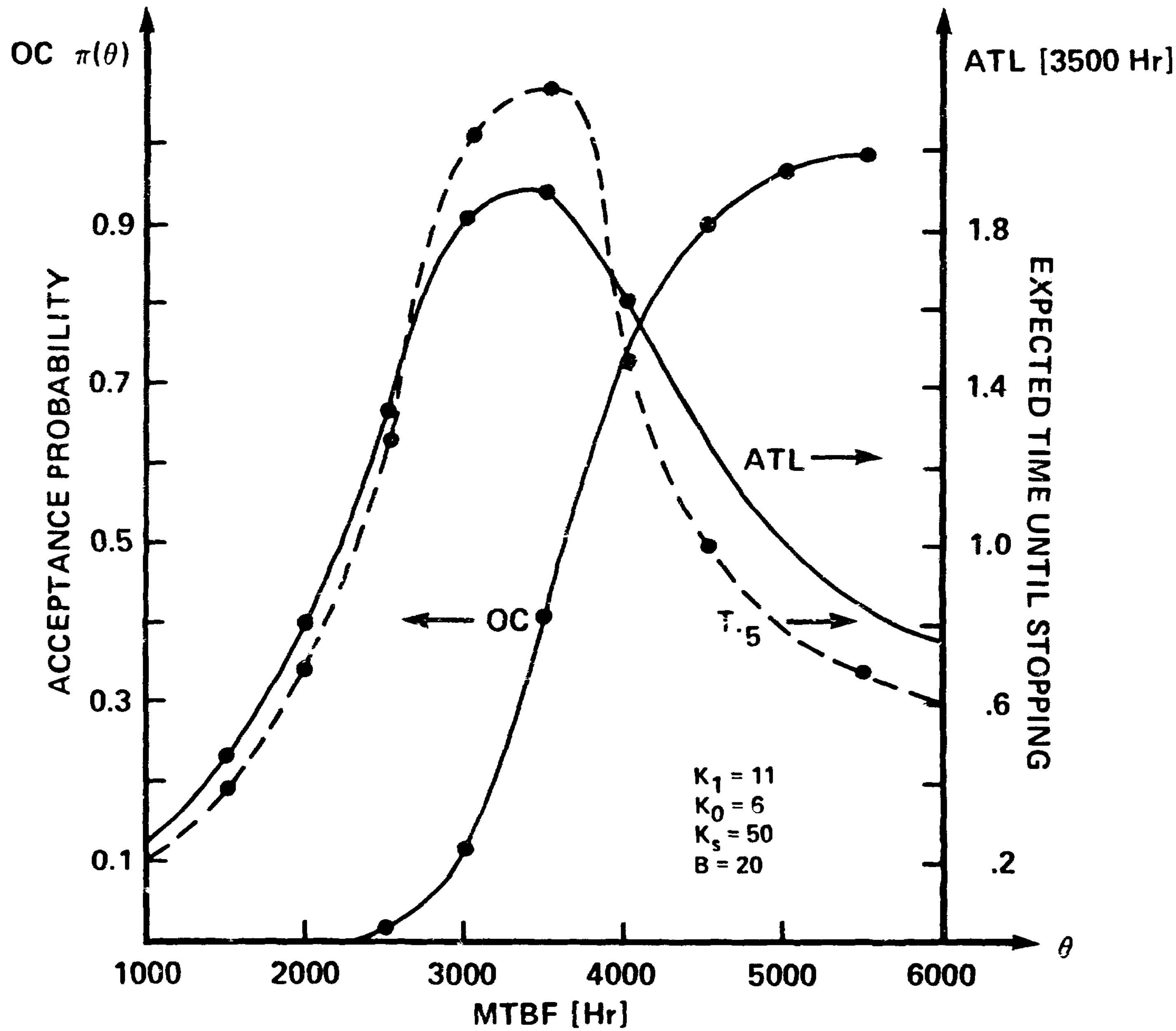
The distributions of τ_R, τ_A and τ,
for $k_1 = 11$, $k_0 = 6$, $k_s = 50$, $b = 20$.

$\theta = 3500$, τ is in units of 3500 hr

τ	$F_\theta^R(\tau)$	$F_\theta^A(\tau)$	$F_\theta(\tau)$
0.05	0.0000		
0.10	0.0000		
0.15	0.0000		
0.20	0.0000		
0.25	0.0001		
0.30	0.0002	0.0025	0.0027
0.35	0.0004	0.0080	0.0084
0.40	0.0008	0.0160	0.0168
0.45	0.0013	0.0260	0.0273
0.50	0.0020	0.0373	0.0394
0.55	0.0029	0.0496	0.0525
0.60	0.0040	0.0623	0.0663
0.65	0.0053	0.0753	0.0806
0.70	0.0068	0.0883	0.0952
0.75	0.0086	0.1013	0.1099
0.80	0.0105	0.1141	0.1246
0.85	0.0126	0.1266	0.1392
0.90	0.0148	0.1389	0.1537
0.95	0.0172	0.1509	0.1681
1.00	0.0198	0.1625	0.1823
1.05	0.0225	0.1738	0.1963
1.10	0.0253	0.1847	0.2100
1.15	0.0282	0.1953	0.2236
1.20	0.0313	0.2056	0.2369
1.25	0.0344	0.2156	0.2500
1.30	0.0376	0.2252	0.2628
1.35	0.0409	0.2346	0.2755
1.40	0.0442	0.2437	0.2878
1.45	0.0476	0.2524	0.3000
1.50	0.0510	0.2610	0.3120
1.55	0.0545	0.2692	0.3237
1.60	0.0579	0.2772	0.3352
1.65	0.0615	0.2850	0.3465
1.70	0.0650	0.2926	0.3576
1.75	0.0686	0.2999	0.3685
1.80	0.0721	0.3070	0.3791
1.85	0.0757	0.3139	0.3896
1.90	0.0793	0.3207	0.3999
1.95	0.0828	0.3272	0.4100
2.00	0.0935	0.3336	0.4271
2.05	0.1111	0.3398	0.4509
2.10	0.1355	0.3458	0.4814
2.15	0.1663	0.3517	0.5180
2.20	0.2026	0.3574	0.5601
2.25	0.2437	0.3630	0.6068
2.30	0.2883	0.3685	0.6568
2.35	0.3350	0.3738	0.7087
2.40	0.3820	0.3790	0.7610
2.45	0.4277	0.3840	0.8118
2.50	0.4704	0.3890	0.8594
2.55	0.5083	0.3938	0.9021
2.60	0.5400	0.3985	0.9385
2.65	0.5643	0.4031	0.9674
2.70	0.5804	0.4077	0.9880
2.75	0.5879	0.4121	1.0000

Figure 2

ACCEPTANCE PROBABILITIES (OC) AND EXPECTED TIME OF TEST (ATL)

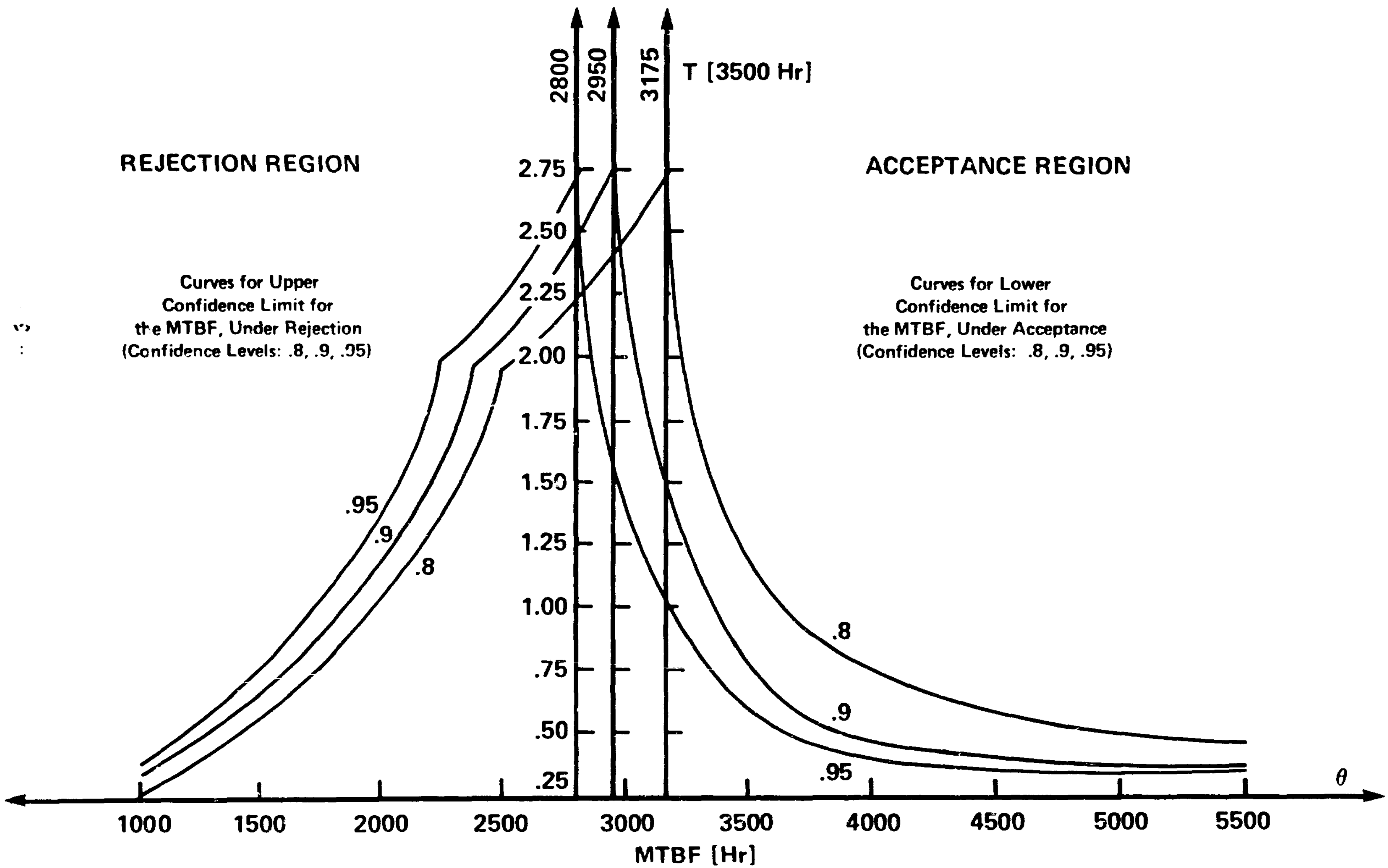


This reflects that the distributions $F_\theta(t)$ are positively skewed in those ranges. On the other hand, if $2{,}650 \le \theta \le 3{,}750$ then $\tau_{.5}(\theta) > ATL(\theta)$, corresponding to negative skewness of $F_\theta(t)$. The graph of $\pi(\theta)$ shows that the probability of accepting a product whose MTBF is less than 3,000 [hr] is not exceeding .1. On the other hand, if the MTBF is greater than 4,500 [hr] then $\pi(\theta) > .9$.

4. Confidence Limits for the MTBF

In Figure 3 we present curves for the determination of the confidence limits for the MTBF, as a function of the stopping time τ. If $\tau = \tau_A$ we use the curves at the acceptance region. The curve designated by $LCL_{.8}$, is actually the graph of the 20-th percentile of the distribution of τ_A, $F_\theta^A(t)$, as a function of θ. We obtain this graph by computing the distribution $F_\theta^A(t)$ at various values of θ greater than 3,175 [hr]. As seen in Figure 2, $\pi(3{,}175) = .2$. Thus, the graph of $\tau_{A,.2}(\theta)$ at $\theta = 3{,}175$ is located at $\tau_A = 2.75$ [3,500 hr], which is the largest value that τ_A can assume. We find the lower confidence limit for θ, at confidence level of $\gamma = .8$, by reverse interpolation in Figure 3. If for example, in an actual life testing we terminate with $\tau_A = .75$ [3,500 hr], we read in Figure 3 the value of θ which yields $\tau_{A,.2}(\theta) = .75$, namely $\theta = 4{,}000$ [hr]. The lower confidence limit is $\underline{\theta}_{.8}(.75) = 4{,}000$ [hr]. If $\tau_A = .5$ [3,500 hr] then $\underline{\theta}_{.8}(.50) = 5{,}000$ [hr]. Notice that the corresponding lower confidence limits for $\gamma = .9$ are read from the graph labeled $LCL_{.9}$ to be $\underline{\theta}_{.9}(.75) = 3{,}500$ [hr] and $\underline{\theta}_{.9}(5) = 3{,}875$ [hr]. If $\tau = \tau_R$, that is the test itself rejects the product, upper confidence intervals for θ, at confidence levels of $\gamma = .8$, .9 and .95 can be determined by using the curves at the rejection region of Figure 3, If, for

Figure 3
CURVES FOR DETERMINING CONFIDENCE LIMITS



example, τ_R = 1.25 [3,500 hr], we read from the graphs that the upper confidence limit at $\gamma = .8$ is $\underline{\theta}_{.8}$ = 2175 hours. If τ_R assumes larger values there is an indication that θ is larger. For example, if τ_R = 2.00 [3,500 hr] then the confidence interval is (2,450, 3,175).

5. Computing the Distributions of Stopping Times

Let $X_n(t)$ designate the total number of failures of the n units on test, during the time interval $(0,t]$. It is assumed that $\{X_n(t); 0 < t\}$ is a homogeneous Poisson process with mean of $\lambda = n/\theta$ failures per time unit, where θ is the MTBF of a single unit. The graphs of $X_n(t)$ are non-decreasing step functions, with jumps of size 1 at the random failure times.

The boundary line $b_R(t)$ assumes integer values on $[0,t^*]$ every Δ units of time, where $\Delta = 1/b$. Accordingly, let $t_i = i\Delta$ $(i = 0,1,2,\ldots,k_0 + k_s)$. The graph of $X_n(t)$ can cross the upper boundary $b_R(t)$ at any time t, but $X_n(t)$ can cross the lower boundary $b_A(t)$ only at the time epochs t_i, $i \geq k_0$. Consider the time interval $(0,t_1]$. $X_n(t)$ could have crossed upper boundary $b_R(t)$ at some point τ, $0 < \tau \leq t \leq t_1$, if and only if $X_n(t) \geq k_1 + 1$. Accordingly, if we denote by $Pos(j;\lambda)$ the CDF of the Poisson distribution with mean λ, at a point j, and by $p(j;\lambda)$ the corresponding probability distribution function (PDF); i.e., $p(j;\lambda) = e^{-\lambda}\lambda^j/j!$, and $Pos(j;\lambda) = \sum_{i=0}^{j} p(i;\lambda)$, then

$$F_\theta(t) = F_\theta^R(t) = 1 - Pos(k_1;\frac{nt}{\theta}),\ 0 < t \leq t_1. \tag{5.1}$$

Define the defective probability distribution function

$$g_\theta(j;t) = P_\theta\{X_n(t) = j,\ \ \tau \geq t\}. \tag{5.2}$$

It follows that, for every $j = 0,\ldots,k_1$,

$$g_\theta(j,t_1) = p(j;\frac{n}{\theta}\, t_1),\ j = 0,\ldots,k_1. \tag{5.3}$$

Furthermore,

$$P_\theta\{\tau > t_1\} = \sum_{j=0}^{k_1} g_\theta(t;t_1) = \mathrm{Pos}(k_1;\ \frac{n}{\theta}\, t_1). \tag{5.4}$$

To derive the distribution for t values greater than t_1 we utilize the property that the Poisson process is an increasing process of <u>independent increments</u>. Thus, for every $2 \le i \le k_0$, we apply the recursive equation

$$g_\theta(j;t_i) = \begin{cases} \displaystyle\sum_{\ell=0}^{j} g_\theta(\ell;t_{i-1})p(j - \ell;\ \frac{n}{\theta}\,\Delta), & 0 \le j \le k_1+i-2 \\ \displaystyle\sum_{\ell=0}^{k_1+i-2} g_\theta(\ell;t_{i-1})p(j - \ell;\frac{n}{\theta}\,\Delta), & j = k_1+i-1 \end{cases} \tag{5.5}$$

Like in (5.4), for every $i = 1,\ldots,k_0$,

$$P_\theta\{\tau > t_i\} = \sum_{j=0}^{k_1+i-1} g_\theta(j;t_i). \tag{5.6}$$

Furthermore, for every $2 \le i \le k_0$,

$$P_\theta\{t_{i-1} < \tau \le t_i\} = P_\theta\{\tau > t_{i-1}\} - P_\theta\{\tau > t_i\} \tag{5.7}$$

$$= \sum_{j=0}^{k_1+i-2} [g_\theta(j;t_{i-1}) - g_\theta(j;t_i)] - g_\theta(k_1 + i - 1;t_1),$$

and

$$F_\theta(t_i) = F_\theta(t_{i-1}) + P_\theta\{t_{i-1} < \tau \le t_i\},\ 1 \le i \le k_0 \tag{5.8}$$

where $t_0 \equiv 0$ and $F_\theta(0) \equiv 0$. Notice that for every $i \leq k_0 - 1$, $F_\theta{}^R(t_i) = F_\theta(t_i)$. However, at $i = k_0$,

$$F_\theta{}^A(t_{k_0}) = g_\theta(0, t_{k_0}) \tag{5.9}$$

and

$$F_\theta{}^R(t_{k_0}) = F_\theta(t_{k_0}) - F_\theta{}^A(t_{k_0}). \tag{5.10}$$

t_{k_0} is the first jump point of $F_\theta{}^A(t)$. The value of $F_\theta(t)$ at any $t_{i-1} < t \leq t_i$, $i = 1, \ldots, k_0$, can be determined from the previous formula by substituting t for t_i in (5.5) - (5.8) and replacing Δ with $(t-t_{i-1})$ in (5.5). The corresponding PDF of $F_\theta(t)$ can be derived from the above formula by differentiation. Alternatively, we can write for $t_{i-1} < t \leq t_i$.

$$\begin{aligned} F_\theta(t) = F_\theta(t_{i-1}) + \sum_{\ell=0}^{k_1+i-2} g_\theta(\ell;\, t_{i-1})[1 - \\ \mathrm{Pos}(k_1 + i - 1 - \ell;\, \tfrac{n}{\theta}(t - t_{i-1}))]. \end{aligned} \tag{5.11}$$

Differentiation with respect to t yields the PDF, for $t_{i-1} < t \leq t_i$,

$$f_\theta(t) = \begin{cases} \frac{n}{\theta} \sum_{\ell=0}^{k_1+i-2} g_\theta(\ell;\, t_{i-1}) p(k_1 + i - 1 - \ell;\, \frac{n}{\theta}(t - t_{i-1})), \\ \qquad\qquad \text{if } i = 2, \ldots, k_0. \\ \frac{n}{\theta}\, p(k_1;\, \frac{n}{\theta}\, t), \qquad \text{if } i = 1. \end{cases} \tag{5.12}$$

We assumed that $k_s \geq k_0 + k_1$. Consider now the time interval $t_{k_0} \leq t \leq t_{k_s - k_1}$. In this interval the two boundary lines $b_R(t)$ and $b_A(t)$ are parallel. On this interval we define for every $1 \leq i \leq k_s - k_0 - k_1$,

$$g_\theta(j;t_{k_0+1}) = \tag{5.13}$$

$$\begin{cases} \sum_{\ell=1}^{j} g(\ell;t_{k_0+i-1})p(j-\ell;\frac{n}{\theta}\Delta), & i \le j \le k_0+k_1+i-2 \\ \sum_{\ell=1}^{k_0+k_1+i-2} g(\ell;\ t_{k_0+i-1})p(k_0+k_1+i-1-\ell;\frac{n}{\theta}\Delta), & j=k_0+k_1+i-1 \end{cases}$$

The probability of acceptance at t_{k_0+1} is

$$P_\theta\{\tau_A = t_{k_0+i}\} = g_\theta(i;t_{k_0+1}), \tag{5.14}$$

and

$$F_\theta^{\,A}(t_{k_0+1}) = F_\theta^{\,A}(t_{k_0+i-1}) + g_\theta(i;t_{k_0+i}). \tag{5.15}$$

As before, we can write,

$$P_\theta\{\tau > t_{k_0+i}\} = \sum_{j=i+1}^{k_0+k_1+i-1} g(j;t_{k_0+i}), \tag{5.16}$$

and hence

$$\begin{aligned} P_\theta\{t_{k_0+i-1} < \tau \le t_{k_0+i}\} &= \sum_{j=i}^{k_0+k_1+i-2} g_\theta(j;t_{k_0+i-1}) \\ &- \sum_{j=i+1}^{k_0+k_1+i-1} g_\theta(j;t_{k_0+i}) = g_\theta(i;t_{k_0+i-1}) \\ &+ \sum_{j=i+1}^{k_0+k_1+i-2} [g_\theta(j;t_{k_0+i-1}) - g_\theta(j;t_{k_0+i})] \\ &- g_\theta(k_0+k_1+i-1;t_{k_0+i}). \end{aligned} \tag{5.17}$$

But since $P_\theta\{\tau_A = t_{k_0+i}\} = g_\theta(i;t_{k_0+i})$,

$$P_\theta\{t_{k_0+i-1} < \tau_R < t_{k_0+i}\} \tag{5.18}$$

$$= \sum_{j=i}^{k_0+k_1+i-2} [g_\theta(j;t_{k_0+i-1}) - g_\theta(j;t_{k_0+i})]$$

$$-g_\theta(k_0 + k_1 + i - 1;t_{k_0+i}).$$

Adding this increment to the value of $F_\theta{}^R(t)$ at t_{k_0+i-1} we obtain the value of $F_\theta{}^R(t)$ at t_{k_0+i}. We can also write, for every $t_{k_0+i-1} < t < t_{k_0+i}$,

$$F_\theta{}^R(t) = F_\theta{}^R(t_{k_0+i-1}) +$$

$$\sum_{\ell=i}^{k_0+k_1+i-2} g_\theta(\ell;t_{k_0+i-1})[1 - \mathrm{Pos}(k_0 + k_1 + i - 1 - \ell; \frac{n}{\theta} (t - t_{k_0+i-1}))]. \tag{5.19}$$

This formula yields by proper differentiation the PDF, for $t_{k_0+i-1} < t < t_{k_0+i}$,

$$f_\theta(t) = \frac{n}{\theta} \sum_{\ell=i}^{k_0+k_1+i-2} g_\theta(\ell;t_{k_0+i-1}) \cdot$$

$$\cdot p(k_0 + k_1 + i - 1 - \ell; \frac{n}{\theta} (t - t_{k_0+i-1})). \tag{5.20}$$

Finally we consider the interval $t_{k_s-k_1} \le t \le t_{k_s+k_0-1}$. On this interval the upper boundary $b_R(t)$ is the constant k_s. The functions $g_\theta(j;t_{k_s-k_1+i})$ are determined recursively, according to the formula, for every $k_s - k_1 - k_0 + i \le j \le k_s - 1$,

$$g_\theta(j;t_{k_s-k_1+i}) = \sum_{\ell=k_s-k_1-k_0+i}^{j} g_\theta(\ell;t_{k_s-k_1+i-1})p(j-\ell;\tfrac{n}{\theta}\Delta), \tag{5.21}$$

$$1 \le i \le k_0 + k_1 - 1.$$

From this one obtains immediately that

$$P_\theta\{\tau_A = t_{k_s-k_1+i}\} = g_\theta(k_s - k_0 - k_1 + i;t_{k_s-k_1+i}), \tag{5.22}$$

and

$$\begin{aligned} &P_\theta\{t_{k_s-k_1+i-1} < \tau \le t_{k_s-k_1+i}\} \\ &= g_\theta(k_s - k_0 - k_1 + i;t_{k_s-k_1+i-1}) \\ &+ \sum_{\ell=k_s-k_0-k_1+i+1}^{k_s-1} [g_\theta(\ell;t_{k_s-k_1+i-1}) - g_\theta(\ell;t_{k_s-k_1+i})]. \end{aligned} \tag{5.23}$$

Accordingly,

$$\begin{aligned} &F_\theta^A(t_{k_s-k_1+i}) = F_\theta^A(t_{k_s-k_1+i-1}) + \\ &g_\theta(k_s - k_1 - k_0 + i;t_{k_s-k_1+i}), \end{aligned} \tag{5.24}$$

and

$$\begin{aligned} &F_\theta^R(t_{k_s-k_1+i}) = F_\theta^R(t_{k_s-k_1+i-1}) + \\ &\sum_{\ell=k_s-k_1-k_0+i}^{k_s-1} [g_\theta(\ell;t_{k_s-k_1+i-1}) - g_\theta(\ell;t_{k_s-k_1+i})]. \end{aligned} \tag{5.25}$$

For values of t in the interval $(t_{k_s-k_1+i-1}, t_{k_s-k_1+i})$ we obtain

$$F_\theta^R(t) = F_\theta^R(t_{k_s-k_1+i-1}) + \sum_{\ell=k_s-k_1-k_0+i}^{k_s-1} g_\theta(\ell; t_{k_s-k_1+i-1})[1 - \mathrm{Pos}(k_s - 1 - \ell; \tfrac{n}{\theta}(t - t_{k_s-k_1+i-1}))], \tag{5.26}$$

and

$$f_\theta(t) = \frac{n}{\theta} \sum_{\ell=k_s-k_1-k_0+i}^{k_s-1} g_\theta(\ell; t_{k_s-k_1+i-1}) \cdot p(k_s - 1 - \ell; \tfrac{n}{\theta}(t - t_{k_s-k_1+i-1})). \tag{5.27}$$

6. The Expected Stopping Time (ATL)

The expected stopping time, or average test length (ATL) is given by

$$\begin{aligned} E_\theta\{\tau\} &= \int_0^{t_{k_0+k_s}} t\,dF_\theta(t) \\ &= \sum_{i=0}^{k_s-1} t_{k_0+i}\, P_\theta\{\tau_A = t_{k_0+i}\} \\ &+ \sum_{\nu=1}^{k_0+k_s-1} \int_{t_{\nu-1}}^{t_\nu} t\,dF_\theta^R(t). \end{aligned} \tag{6.1}$$

The first term on the r.h.s. of (6.1) is equal to

$$E_\theta^A\{\tau\} = \sum_{i=0}^{k_s-1} t_{k_0+i}\, g_\theta(i; t_{k_0+i}). \tag{6.2}$$

The second term on the r.h.s. of (6.1) is expressed as a sum of three components, $E^R_{1,\theta}$, $E^R_{2,\theta}$ and $E^R_{3,\theta}$, where

$$E^R_{1,\theta} = \sum_{\nu=1}^{k_0} \int_{t_{\nu-1}}^{t_\nu} t dF_\theta{}^R(t),$$

$$E^R_{2,\theta} = \sum_{\nu=k_0+1}^{k_s-k_1} \int_{t_{\nu-1}}^{t_\nu} t dF_\theta{}^R(t), \tag{6.3}$$

and

$$E^R_{3,\theta} = \sum_{\nu=k_s-k_1+1}^{k_s+k_0-1} \int_{t_{\nu-1}}^{t_\nu} t dF_\theta{}^R(t). \tag{6.4}$$

The following result is applied in the development of the formulae for $E_i{}^R(i = 1,3)$; namely,

$$\int_{t_{\nu-1}}^{t_\nu} tp(j;\lambda(t - t_{\nu-1}))dt = \\ = \frac{t_{\nu-1}}{\lambda} (j + 1)(1 - \mathrm{Pos}(j;\lambda\Delta)) + \frac{(j+1)(j+2)}{\lambda^2} (1 - \mathrm{Pos}(j + 1;\lambda\Delta)). \tag{6.5}$$

According to (5.12) and (6.5),

$$E^R_{1,\theta} = \sum_{i=1}^{k_0} \sum_{\ell=0}^{k_1+1-2} g_\theta(\ell;t_{i-1})\{t_{i-1}(k_1 + i - \ell)\cdot \\ \cdot(1 - \mathrm{Pos}(k_1 + i - \ell - 1; \frac{n}{\theta} \Delta)) + \frac{\theta}{n} (k_1 + i - \ell)(k_1 + i + 1 - \ell)\cdot \\ \cdot(1 - \mathrm{Pos}(k_1 + i - \ell; \frac{n}{\theta}))\}. \tag{6.6}$$

For $E^R_{2,\theta}$ we use (5.20) and (6.5) and obtain the formula:

$$E^R_{2,\theta} = \sum_{i=1}^{k_s-k_1-k_0} \sum_{\ell=i}^{k_0+k_1+i-2} g_\theta(\ell;\, t_{k_0+i-1}) \cdot$$

$$\{t_{k_0+i-1}(k_0 + k_1 + i - \ell)[1 - \mathrm{Pos}(k_0 + k_1 + i - 1 - \ell;\, \tfrac{n}{\theta}\Delta)] \tag{6.7}$$

$$+ \frac{\theta}{n}(k_0 + k_1 + i - \ell)(k_0 + k_1 + i + 1 - \ell)[1 - \mathrm{Pos}(k_0 + k_1 + i - \ell;\tfrac{n}{\theta}\Delta)]\}.$$

Finally, from (5.27) and (6.5) we obtain

$$E^R_{3,\theta} = \sum_{i=1}^{k_0+k_1-1} \sum_{\ell=k_s-k_1-k_0+i}^{k_s-1} g_\theta(\ell; t_{k_s-k_1+i-1}) \cdot$$

$$\{t_{k_s-k_1+i-1}(k_s - \ell)[1 - \mathrm{Pos}(k_s - 1 - \ell;\tfrac{n}{\theta}\Delta)] +$$

$$\frac{\theta}{n} \cdot (k_s - \ell)(k_s + 1 - \ell)[1 - \mathrm{Pos}(k_s - \ell;\, \tfrac{n}{\theta}\Delta)]\}.$$

Formulae (6.2), (6.6), (6.7) and (6.8) yield the expected stopping time.

7. Concluding Remarks

As shown in the previous sections, in order to determine a lower confidence limit for the MTBF, θ, after stopping according to the rules of sequential life testing, one has to graph the percentile points $\tau_{A,\alpha}(\theta)$, of the distribution $F_\theta^A(t)$ as a function of θ. This requires the tabulation of the distributions $F_\theta^A(t)$, $F_\theta^R(t)$ and $F_\theta(t)$, for various values of θ. In the Appendix we provide a BASIC program for the computation of the cumulative distributions $F_\theta^R(t)$, $F_\theta^A(t)$ and $F_\theta(t)$ for various values of θ. After compilation, it has taken about five minutes to run this program on an IBM PC, with five different values of θ.

By changing the parameters k_0, k_1, k_s and b of the stopping boundaries one changes the characteristics of the procedure. Thus, if it is desired to change the acceptance probabilities $\pi(\theta)$ or the average test length, ATL(θ), it is possible to try different values of the boundary line parameters and examine the resulting graphs of $\pi(\theta)$ and ATL(θ).

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APPENDIX

BASIC PROGRAM FOR THE COMPUTATION OF THE DISTRIBUTIONS OF STOPPING TIMES

```
10 PRINT "  A PROGRAM FOR COMPUTING THE CHARACTERISTICS OF THE DESIGN"
20 PRINT "  insert values of n,k0,k1,ks,b in order"
30 INPUT N,K0,K1,KS,B
40 KB=KS-K0-K1
50 KBP=KB+1:KBM=KB-1
60 K1P=K1+1:K1M=K1-1
65 KSP=KS+1:KSM=KS-1
70 DLT=1/B
75 DIM H(KSP),G(KSP),PT(KS),PC(KS+K0)
80 TET1=3500
90 FOR TET=1500 TO 5500 STEP 1000
100 W=TET/TET1
110 TO=K0*DLT
120 LPRINT USING"  ##   ##   ##   ##   ##.#   ####";N,K0,K1,KS,B,TET
130 HET=DLT*N/W
135 LAM=HET
140 FOR I=0 TO KS
150 IF I> K1 THEN GOTO 230
170 M=1
180 GOSUB 1300
185 FI=PS
190 M=M-1
200 GOSUB 1300
210 H(I)=FI-PS
220 GOTO 240
230 H(I)=0
240 NEXT I
250 M=K1
260 GOSUB 1300
270 CDF1=1-PS
280 PC(1)=CDF1
290 T1=DLT
300 LPRINT USING "  #.##    #.####"; T1,CDF1
310 FOR I=2 TO K0
320 KI=K1+I
330 KIP=KI+1:KIM=KI-1
340 FOR J=0 TO KIM
350 SUMJ=0
360 FOR L=0 TO J
370 JL=J-L
380 M=JL: GOSUB 1300
390 FI=PS
400 M=M-1: GOSUB 1300
410 SUMJ=SUMJ+H(L)*(FI-PS)
420 NEXT L
```

```
450 G(J)=SUMJ
460 NEXT J
470 FOR J=KI TO KS
480 G(J)=0: NEXT J
490 TI=I*DLT
500 TUM=0
510 FOR L=0 TO (KIM-1)
520 TUM=TUM+H(L)-G(L)
530 NEXT L
540 CDF1=CDF1+TUM-G(KIM)
550 PC(I)=CDF1
555 IF I=KO THEN GOTO 570
560 LPRINT USING"  #.##   #.#### ";TI,CDF1
570 FOR J=0 TO KS
580 H(J)=G(J)
590 NEXT J:NEXT I
600 PT(0)=H(0):CDF2=PT(0)
610 CDFT=CDF1+CDF2
620 LPRINT USING "  #.##   #.####   #.####   #.####";TI,CDF1,CDF2,CDFT
630 FOR I=1 TO KB
640 FOR J=0 TO I
650 IF J<I THEN G(J)=0
660 NEXT J
670 KI=KO+K1+I: KIM=KI-1
680 FOR J=I TO KIM
690 SUMJ=0
700 FOR L=I TO J
710 JL=J-L
720 M=JL: GOSUB 1300
730 PI=PS
740 M=M-1 :GOSUB 1300
750 SUMJ=SUMJ+H(L)*(PI-PS)
760 NEXT L
770 G(J)=SUMJ
775 NEXT J
780 PT(I)=G(I)
790 CDF2=CDF2+PT(I)
800 TI=TI+DLT
810 TUM=0
820 FOR L=I TO (KIM-1)
830 IF L=I THEN TUM=TUM+H(L) ELSE TUM=TUM+H(L)-G(L)
840 NEXT L
850 CDF1=CDF1+TUM-G(KIM)-PT(I)
860 PC(KO+I)=CDF1: CDFT=CDF1+CDF2
870 LPRINT USING "  #.##   #.####   #.####   #.####";TI,CDF1,CDF2,CDFT
880 FOR J=0 TO KS
890 H(J)=G(J): NEXT J:NEXT I
900 FOR I=KBP TO KSM
```

```
910 FOR J=0 TO (I-1)
920 G(J)=0 :NEXT J
930 FOR J=I TO KSM
940 SUMJ=0
950 FOR L=I TO J
960 JL=J-L
970 M=JL: GOSUB 1300
980 PI=PS: M=M-1: GOSUB 1300
990 SUMJ=SUMJ+H(L)*(PI-PS)
1000 NEXT L
1010 G(J)=SUMJ :NEXT J
1020 PT(I)=G(I)
1030 CDF2=CDF2+PT(I)
1040 TI=TI+DLT
1050 TUM=0
1060 FOR L=I TO KSM
1070 IF L=I THEN TUM=TUM+H(L) ELSE TUM=TUM+H(L)-G(L)
1080 NEXT L
1090 CDF1=CDF1+TUM-PT(I)
1100 PC(K0+I)=CDF1: CDFT=CDF1+CDF2
1110 LPRINT USING "  #.##   #.####   #.####   #.####";TI,CDF1,CDF2,CDFT
1120 FOR J=0 TO KS
1130 H(J)=G(J):NEXT J:NEXT I
1140 ASN=0
1150 KSV=KS+K0
1160 FOR I=1 TO (KSV-1)
1170 TI=I*DLT: LI=1-K0
1180 IF I>= K0 THEN ASN=ASN+TI*PT(LI)
1190 IF I=1 THEN PF=PC(1) ELSE PF=PC(I)-PC(I-1)
1200 ASN=ASN+(TI-DLT/2)*PF
1210 NEXT I
1220 OCP=CDF2
1230 LPRINT USING "    ocp=#.####   asn=#.##";OCP,ASN
1235 NEXT TET
1250 END
1300 IF M<0 THEN PS=0
1310 IF M<0 THEN GOTO 1400
1320 PR=EXP(-LAM)
1330 IF M=0 THEN PS=PR
1350 IF M=0 THEN GOTO 1400
1355 CP=PR
1360 FOR KP=1 TO M
1370 PR=PR*LAM/KP
1380 CP=CP+PR
1390 NEXT KP
1395 PS=CP
1400 RETURN
```

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